p-Hyponormality of weighted composition operators (Inequalities on Linear Operators and its Applications)

Cho, Muneo; Yamazaki, Takeaki

数理解析研究所講究録
第 1596 巻 2008 年 18-24

p-Hyponormality of weighted composition operators

神奈川大学　長　宗雄 (Muneo Chō)
Kanagawa University
神奈川大学　山崎 丈明 (Takeaki Yamazaki)
Kanagawa University

ABSTRACT

Characterizations of the classes of p-hyponormal, ∞-hyponormal and weak hyponormal weighted composition operators are introduced. It is shown that some classes of weak hyponormal weighted composition operators can not be separated. It is an extension of the result by C. Burnap, etc.

This report is based on the following paper:
[CY] M. Chō and T. Yamazaki, *Characterizations of p-hyponormal and weak hyponormal weighted composition operators*, preprint.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space, and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For a bounded linear operator, many authors have studied properties of weak normality of operators, especially, the class of hyponormal operators defined as follows:

Definition 1. Let $T \in B(\mathcal{H})$. Then the following operator classes are defined:

(i) T is hyponormal $\Longleftrightarrow T^*T \geq TT^*$,

(ii) for $p > 0$, T is p-hyponormal $\Longleftrightarrow (T^*T)^p \geq (TT^*)^p$.

Especially, if T is p-hyponormal for all $p > 0$, we call T ∞-hyponormal ([10]).

It is well known that typical examples of these operators are expressed by polynomials of weighted shift operators on l^2. So many authors have studied properties of weighted shift operators. Examples are obtained as follows:

Example 1.1. Let U be a weighted unilateral shift on l^2 as follows:

$$U = \begin{pmatrix} 0 & 0 & & & \\ 1 & 0 & 0 & & \\ & 2 & 0 & 0 & \\ & & 2 & 0 & 0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then

(i) U is ∞-hyponormal, not quasinormal (i.e., $T^*TT \neq TT^*T$),
(ii) $2U + U^*$ is hyponormal, not ∞-hyponormal,
(iii) $(2U + U^*)^2$ is $\frac{1}{2}$-hyponormal, not hyponormal.

Moreover, classes of weak hyponormal operators have been also defined, especially, the following operator classes are studied by many authors, strenuously.

Definition 2. Let $T \in B(\mathcal{H})$. Then the following operator classes are defined:
(i) T is quasihyponormal $\Longleftrightarrow T^*T^*TT \geq T^*TT^*T$,
(ii) for $p > 0$, T is p-quasihyponormal $\Longleftrightarrow T^*(T^*T)^pT \geq T^*(TT^*)^pT$,
(iii) T belongs to class A $\Longleftrightarrow |T^2| \geq |T|^2$ ([4, 5]),
(iv) for $s, t > 0$, T belongs to class $\mathrm{A}(s,t)$
$$\Longleftrightarrow (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \text{ ([3]).}$$

We remark that the class A coincides with the class $\mathrm{A}(1,1)$. These operator classes are defined by operator inequalities. As parallel classes of them, the classes of paranormal and absolute-(s,t)-paranormal operators are known.

Definition 3. Let $T \in B(\mathcal{H})$. Then the following operator classes are defined:
(i) T is paranormal $\Longleftrightarrow \|T^2x\|\|x\| \geq \|Tx\|^2$ for all $x \in \mathcal{H}$ ([4]),
(ii) for $s, t > 0$, T is absolute-(s,t)-paranormal
$$\Longleftrightarrow \||T|^s|T^*|^tx\|^t\|x\|^s \geq \||T^*|^tx\|^{s+t} \text{ for all } x \in \mathcal{H} \text{ ([12]).}$$

We also remark that the classes of paranormal and absolute-$(1,1)$-paranormal operators are the same. These operator classes are defined by norm inequalities. Inclusion relations among above operator classes are well known [12] as follows: For a fixed $p > 0$,

$$\begin{aligned}\{p-\text{hyponormal}\} &\subset \{p-\text{quasihyponormal}\} \\ &\subset \text{class } \mathrm{A}(p,1) \subset \{\text{absolute}-(p,1)-\text{paranormal}\}.\end{aligned} \tag{1.1}$$

The above inclusion relations are all proper. To study these operators, weighted shift operators are very usefull tools. As an extension of weighted shift, weighted composition operators (it's definition will be introduced in the below) are known. Hence, to study some operator classes related to hyponormal operators, it is better that we know properties of weighted composition operators. Study of hyponormal weighted composition operators has been started by A. Lambert in [7]. Recently, weak hyponormal composition operators are studied in [2, 1], they have shown some characterizations of weak hyponormal composition operators, and obtained concrete examples for related hyponormal composition operators. But they discussed on composition operators (not weighted composition operators), mainly.

In this report, we shall obtain some characterizations of related hyponormal weighted composition operators. In section 2, we shall prepare the definition and basic properties of weighted composition operators. In section 3, we shall discuss a characterization of p and ∞-hyponormalities of weighted composition operators. They are extensions of the results in [7] and [2]. In section 4, we shall show that the classes of p-quasihyponormal and absolute-$(p,1)$-paranormal weighted composition operators are the same. This is an extension of the results in [1].

2. Definition and basic properties of weighted composition operators

In this section, we shall introduce the definition and basic properties of weighted composition operators.

Definition 4. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. A measurable transformation $T : X \to X$ with $T^{-1}\mathcal{F} \subseteq \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. For a non negative $w \in L^{\infty}(X, \mathcal{F}, \mu)$, define the weighted composition operator C on $L^2(X, \mathcal{F}, \mu)$ as

$$Cf = wf \circ T \quad \text{for } f \in L^2(X, \mathcal{F}, \mu).$$

Especially, the case $w \equiv 1$, we call C a composition operator, simply.

In the case $h = \dfrac{d\mu \circ T^{-1}}{d\mu} \in L^{\infty}$, C is bounded. We can consider that weighted composition operators are kind of shift operators.

Example 2.1. Let $X = \mathbb{N}$, a transformation $T(n)$ be

$$T(n) = \begin{cases} 1 & (n = 1) \\ n-1 & (n \geq 2) \end{cases}$$

and $w = (0, 1, 2, 2, \cdots) \in l^{\infty}$. Then for $f = (f_1, f_2, \cdots) \in l^2$,

$$Cf = wf \circ T = w(f_1, f_1, f_2, f_3, \cdots) = (0, f_1, 2f_2, 2f_3, \cdots),$$

i.e., C is weighted shift which is the same as U in Example 1.1.

Let $Ef = E(f|T^{-1}\mathcal{F})$ be the conditional expectation of f with respect to $T^{-1}\mathcal{F}$. Ef derived its uses from the idea that it represents f on the average with respect to $T^{-1}\mathcal{F}$. Specifically, for each $A \in T^{-1}\mathcal{F}$, $\int_A f d\mu = \int_A Ef d\mu$. This means that except when f is $T^{-1}\mathcal{F}$-measurable, Ef and f are never related by a pointwise inequality, and conditional expectation is of limited value in making pointwise estimates to the value of a function.

Example 2.2. Let $X = \mathbb{N}$, a transfornation $T(n)$ be

$$T(n) = \begin{cases} 1 & (n = 1) \\ n-1 & (n \geq 2) \end{cases}.$$

Then $T^{-1}\mathcal{F}$ is generated by the atoms

$$\{1, 2\},\ \{3\},\ \{4\}, \cdots.$$

Moreover the measure μ is defined by $\mu(\{n\}) = 1$ for $n \in \mathbb{N}$. Hence for $f = (f_1, f_2, f_3, \cdots)$,

$$\begin{aligned} Ef &= \left(\frac{\mu(\{1\})f_1 + \mu(\{2\})f_2}{\mu(\{1,2\})}, \frac{\mu(\{1\})f_1 + \mu(\{2\})f_2}{\mu(\{1,2\})}, \frac{\mu(\{3\})f_3}{\mu(\{3\})}, \frac{\mu(\{4\})f_4}{\mu(\{4\})}, \cdots\right) \\ &= \left(\frac{f_1 + f_2}{2}, \frac{f_1 + f_2}{2}, f_3, f_4, \cdots\right). \end{aligned}$$

To study weighted composition operators, we prepare some important properties of conditional expectation as follows: First of all we note that for any nonnegative function f, $\mathrm{supp}(Ef)$ is the smallest (up to null sets) $T^{-1}\mathcal{F}$ set containing $\mathrm{supp}(f)$ in [9], i.e.,

$$\mathrm{supp}(f) \subseteq \mathrm{supp}(Ef)$$

always holds. Hence, $\dfrac{f}{Ef}$ is well defined.

Lemma A. *Let $C = U|C|$ be the polar decomposition of weighted composition operator C and $h = \dfrac{d\mu \circ T^{-1}}{d\mu}$. Then for $f \in L^2(X, \mathcal{F}, \mu)$, the following hold:*

(i) $C^*f = hE(wf) \circ T^{-1}$ [7],

(ii) $C^*Cf = hEw^2 \circ T^{-1}f$.

In what follows, we write the modulus of C by a, i.e., $C^*Cf = hEw^2 \circ T^{-1}f = a^2 f$. Moreover by $h \circ T > 0$ [6], the partial isometry part U of C can be expressed as follows:

$$Uf = \frac{w}{a \circ T} f \circ T.$$

Lemma B ([11]). *The conditional expectation E is a projection onto $T^{-1}\mathcal{F}$, i.e., if $g, k \in L^2(X, \mathcal{F}, \mu)$, then there exists $G \in L^2(X, \mathcal{F}, \mu)$ such that $Eg = G \circ T$. Hence $E(g \circ T \cdot k) = g \circ T \cdot Ek$ and $E(Eg \cdot k) = Eg \cdot Ek$ hold.*

More properties are listed in [2].

3. Characterizations of p and ∞-hyponormal operators

In this section, we shall introduce characterizations of p and ∞-hyponormal weighted composition operators.

Theorem 3.1. *For $p > 0$, a weighted composition operator C is p-hyponormal if and only if the following conditions hold:*

(i) $\mathrm{supp}(w) \subseteq \mathrm{supp}(a)$,

(ii) $E\left[\left(\dfrac{a \circ T}{a}\right)^{2p} \dfrac{w^2}{Ew^2}\right] \leq 1$ *a.e.*

The case $w \equiv 1$ has been shown in [2]. To prove Theorem 3.1, we use the following characterization of hyponormal weighted composition operators:

Theorem C ([7]). *A weighted composition operator C is hyponormal if and only if the following conditions hold:*

(i) $\mathrm{supp}(w) \subseteq \mathrm{supp}(a)$,

(ii) $E\left[\left(\dfrac{a \circ T}{a}\right)^{2} \dfrac{w^2}{Ew^2}\right] \leq 1$ *a.e.*

Proof of Theorem 3.1. Let $C = U|C|$ be the polar decomposition of C. C is p-hyponormal if and only if $C_p = U|C|^p$ is hyponormal. C_p is also a weighted composition operator as follows:

$$C_p f = U|C|^p f = Ua^p f = \frac{w}{a \circ T} a^p \circ T f \circ T = wa^{p-1} \circ T f \circ T = w_p f \circ T,$$

where $w_p \equiv wa^{p-1} \circ T$. Then by Lemma B,

$$Ew_p^2 = E(w^2 \cdot a^{2(p-1)} \circ T) = Ew^2 \cdot a^{2(p-1)} \circ T. \tag{3.1}$$

Moreover we have $|C_p|f = |C|^p f = a^p f$. Hence by Theorem C, we have only to prove that

(i) $\operatorname{supp}(w_p) \subseteq \operatorname{supp}(a^p) \iff \operatorname{supp}(w) \subseteq \operatorname{supp}(a)$,

(ii) $E\left[\left(\frac{a^p \circ T}{a^p}\right)^2 \frac{w_p^2}{Ew_p^2}\right] \leq 1 \iff E\left[\left(\frac{a \circ T}{a}\right)^{2p} \frac{w^2}{Ew^2}\right] \leq 1,$

see [CY]. □

For each non-negative $f \in L^2(X, \mathcal{F}, \mu)$, $(Ef^p)^{\frac{1}{p}}$ is increasing on $p > 0$ [8] by Hölder's inequality and $(Ef^p)^{\frac{1}{p}} \leq \|f\|_\infty < +\infty$. Then there exists $M(f) = s - \lim_{p \to \infty} (Ef^p)^{\frac{1}{p}}$ and we call it minimal majorant of f. It is known that $f \leq M(f)$ holds in [8].

Next we will show a characterization of ∞-hyponormal weighted composition operators as follows:

Theorem 3.2. *A weighted composition operator C is ∞-hyponormal if and only if the following conditions hold:*

(i) $\operatorname{supp}(w) \subseteq \operatorname{supp}(a)$,

(ii) $a \circ T \leq a$ *on* $\chi_{\operatorname{supp}(w)}$,

where χ_N means the characteristic function on N.

To prove Theorem 3.2, we shall prepare the following lemma:

Lemma 3.3. *Let $a, b \in L^2(X, \mathcal{F}, \mu)$ with $a, b \geq 0$. Then*

$$s\text{-}\lim_{p \to \infty} \{E(a^p b)\}^{\frac{1}{p}} = M(a\chi_{\operatorname{supp}(b)}).$$

Proof is given in [CY].

Proof of Theorem 3.2. By the definition of ∞-hyponormality of C, C is p-hyponormal for all $p > 0$, that is, $\operatorname{supp}(w) \subseteq \operatorname{supp}(a)$ and

$$E\left[\left(\frac{a \circ T}{a}\right)^{2p} \frac{w^2}{Ew^2}\right] \leq 1$$

hold for all $p > 0$ by Theorem 3.1. Then by Lemma 3.3 and $f \leq M(f)$,

$$\begin{aligned} 1 &\geq s - \lim_{p \to \infty} \left(E\left[\left(\frac{a \circ T}{a}\right)^{2p} \frac{w^2}{Ew^2}\right]\right)^{\frac{1}{p}} \\ &= M\left(\left(\frac{a \circ T}{a}\right)^2 \cdot \chi_{\operatorname{supp}(\frac{w^2}{Ew^2})}\right) \geq \left(\frac{a \circ T}{a}\right)^2 \cdot \chi_{\operatorname{supp}(\frac{w^2}{Ew^2})}. \end{aligned}$$

Here, by $\operatorname{supp}(w^2) \subseteq \operatorname{supp}(Ew^2)$, we have $a \circ T\chi_{\operatorname{supp}(w)} \leq a$. Moreover, by $\operatorname{supp}(w) \subseteq \operatorname{supp}(a)$ we have (ii).

Conversely, if $a \circ T \leq a$ on $\mathrm{supp}(w)$ holds, then we have

$$1 \geq Ew^2 \cdot \frac{1}{Ew^2} = E\left(\frac{w^2}{Ew^2}\right) \geq E\left[\left(\frac{a \circ T}{a} \cdot \chi_{\mathrm{supp}(w)}\right)^{2p} \frac{w^2}{Ew^2}\right] = E\left[\left(\frac{a \circ T}{a}\right)^{2p} \frac{w^2}{Ew^2}\right]$$

hold for all $p > 0$, that is, C is ∞-hyponormal. □

4. Equivalent classes

In this section, we shall prove that for a fixed $p > 0$, the class of p-quasihyponormal weighted composition operators coincides with the class of absolute-$(p, 1)$-paranormal weighted composition operators, and obtain their characterization. Hence some weak hyponormal classes in (1.1) are coincide with each other in the weighted composition operators case. We remark that, absolute-$(p, 1)$-paranormal has been introduced in the name absolute-p-paranormal in [5], firstly.

Theorem 4.1. *Let C be a weighted composition operator. For $p > 0$, the following conditions are equivalent:*

(i) *C is p-quasihyponormal,*
(ii) *C belongs to class $A(p, 1)$,*
(iii) *C is absolute-$(p, 1)$-paranormal,*
(iv) $a^{2p} \circ T \cdot Ew^2 \leq E(w^2 a^{2p})$.

The case $w \equiv 1$ has been already shown in [1].

Proof. Inclusions $(i) \implies (ii) \implies (iii)$ have been already known as (1.1). So we have to show the inclusions $(iv) \iff (i)$ and $(iii) \iff (iv)$.

Proof of $(iv) \iff (i)$. C is p-quasihyponormal if and only if

$$C^*(C^*C)^pC \geq C^*(CC^*)^pC.$$

For $f \in L^2(X, \mathcal{F}, \mu)$, by simple calculation (detail is given in [CY]), we have

$$C^*(C^*C)^pCf = hE(w^2a^{2p}) \circ T^{-1}f.$$

On the other hand,

$$C^*(CC^*)^pCf = (C^*C)^{p+1}f = a^{2(p+1)}f.$$

Hence C is p-quasihyponormal if and only if

$$\begin{aligned} a^{2(p+1)} &\leq h(Ew^2a^{2p}) \circ T^{-1} \quad (4.1)\\ \iff a^{2(p+1)} \circ T &\leq h \circ TE(w^2a^{2p}) \\ \iff a^{2p} \circ TEw^2 &\leq E(w^2a^{2p}) \\ \iff (iv). \end{aligned}$$

Proof of $(iii) \iff (iv)$. In [4, Section 3.5.5, Theorem 1], C is absolute-$(p, 1)$-paranormal if and only if

$$C^*|C|^{2p}C - (p+1)\lambda^p|C|^2 + p\lambda^{p+1}I \geq 0 \quad \text{for all } \lambda > 0. \tag{4.2}$$

Here

(1) $C^*|C|^{2p}Cf = hE(w^2a^{2p}) \circ T^{-1}f$,

(2) $|C|^2 f = a^2 f$.

Then (4.2) is equivalent to

$$hE(w^2 a^{2p}) \circ T^{-1} - (p+1)\lambda^p a^2 + p\lambda^{p+1} \geq 0 \quad \text{for all } \lambda > 0.$$

Put $\lambda = a^2$, then it is equivalent to

$$hE(w^2 a^{2p}) \circ T^{-1} \geq a^{2(p+1)},$$

and it is equivalent to (4.1), so does (iv). □

References

[1] C. Burnap and I.B. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl., **337** (2008), 686–694.

[2] C. Burnap, I.B. Jung and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53** (2005), 381–397.

[3] M. Fujii, D. Jung, S.H. Lee, M.Y. Lee and R. Nakamoto, *Some classes of operators related to paranormal and log-hyponormal operators*, Math. Japon., **51** (2000), 395–402.

[4] T. Furuta, *Invitation to linear operators*, Taylor and Francis Inc. London and New York, 2001.

[5] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math., 1 (1998), 257–259.

[6] D. Harrington and R. Whitley, *Seminormal composition operators*, J. Operator Theory, **11** (1984), 125–135.

[7] A. Lambert, *Hyponormal composition operators*, Bull. London Math. Soc., **39** (1986), 395–400.

[8] A. Lambert, *Measurable majorants in L^1*, Glasgow Math. J., **39** (1997), 183–192.

[9] A. Lambert and B. Weinstock, *Descriptions of conditional expectations induced by non-measure-preserving transformations*, Proc. Amer. Math. Soc., **123** (1995), 897–903.

[10] S. Miyajima and I. Saito, ∞ *-Hyponormal operators and their spectral properties*, Acta Sci. Math. (Szeged), **67** (2001), 357–371.

[11] M. Rao, *Conditional measures and applications*, Marcel Decker, New York, 1993.

[12] T. Yamazaki and M. Yanagida, *A further generalization of paranormal operators*, Sci. Math., **3** (2000), 23–32.

Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan
E-mail address: chiyom01@kanagawa-u.ac.jp

Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan
E-mail address: yamazt26@kanagawa-u.ac.jp